MAT123

Logarithms

Inverse of Exponential Function

To find its inverse, first look at the exponent's format:

$$f(x) = b^x$$ go through steps of taking inverse

$$y = b^x$$

works bidirectionally

$$x = b^y$$ how to solve for y?

Use logarithm format: $\boxed{x = b^y \Leftrightarrow y = \log_b x}$

$$\therefore f^{-1}(x) = \log_b x$$

Fact:
Logs and Exponents are inverses of each other

use one to cancel and/or solve other

ex. Write in equivalent exponential form:

- $2 = \log_5 x$

  $y = \log_b x \Rightarrow b^y = x$

  $5^2 = x$ $\quad \boxed{x = 25}$
- $3 = \log_b 64$ $\quad \boxed{b^3 = 64}$
- $y = \log_3 7$ $\quad \boxed{3^y = 7}$

ex. Write in equivalent logarithmic form:

- $12^2 = x$ $\quad \boxed{\log_{12} x = 2}$
- $b^3 = 8$ $\quad \boxed{\log_b 8 = 3}$
- $e^y = 9$ $\quad \log_e 9 = y$ write as $\boxed{\ln 9 = y}$ natural logarithm
- $10^y = 7$ $\quad \log_{10} 7 = y$ write as $\boxed{\log 7 = y}$ common logarithm

Evaluating Logarithms

add $= y$ to create equation

- $\log_2 16$

$$y = \log_2 16$$
$$2^y = 16 \quad \text{since } 2^4 = 16$$
$$\boxed{y = 4}$$

- $\log_7 \frac{1}{49}$

$$y = \log_7 \frac{1}{49}$$
$$7^y = \frac{1}{49}$$
$$7^y = 49^{-1} \text{ get common base of 7}$$
$$7^y = \left(7^2\right)^{-1}$$
$$7^y = 7^{-2}$$
$$\boxed{y = -2}$$

- $\log_{25} 5$

$$y = \log_{25} 5$$
common base $= 5$ $\quad 25^y = 5$
$$\left(5^2\right)^y = 5$$
$$5^{2y} = 5^1$$

once bases are same
just compare exponents

$$2y = 1$$
$$25^{1/2} = \sqrt{25} = 5 \qquad \boxed{y = \frac{1}{2}}$$

- $\log_2 \sqrt[5]{2}$

$$y = \log_2 \sqrt[5]{2}$$
$$2^y = \sqrt[5]{2}$$
$$2^y = 2^{1/5}$$
$$\boxed{y = \frac{1}{5}}$$

<u>Recall Exponent Laws:</u>

$$x^a x^b = x^{a+b}$$
$$\left(x^a\right)^b = x^{ab}$$
$$x^{-a} = \frac{1}{x^a}$$
$$x^a = \frac{1}{x^{-a}}$$
$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

$$\boxed{x = b^y \Leftrightarrow y = \log_b x}$$

Special Logarithmic Properties

also…

$\log_b b = 1$ since $b^1 = b$

ex. Evaluate $\log_7 7 = 1$

$\boxed{\ln e = 1}$

$\log_b 1 = 0$ since $b^0 = 1$

ex. Evaluate $\log_5 1 = 0$

$\boxed{\ln 1 = 0}$

<u>Inverse Properties:</u>

$(f \circ g)(x) = x$

$(g \circ f)(x) = x$

$\log_b b^x = x$

$b^{\log_b x} = x$

ex. Evaluate $\cancel{\log_7 7}^5 = 5$

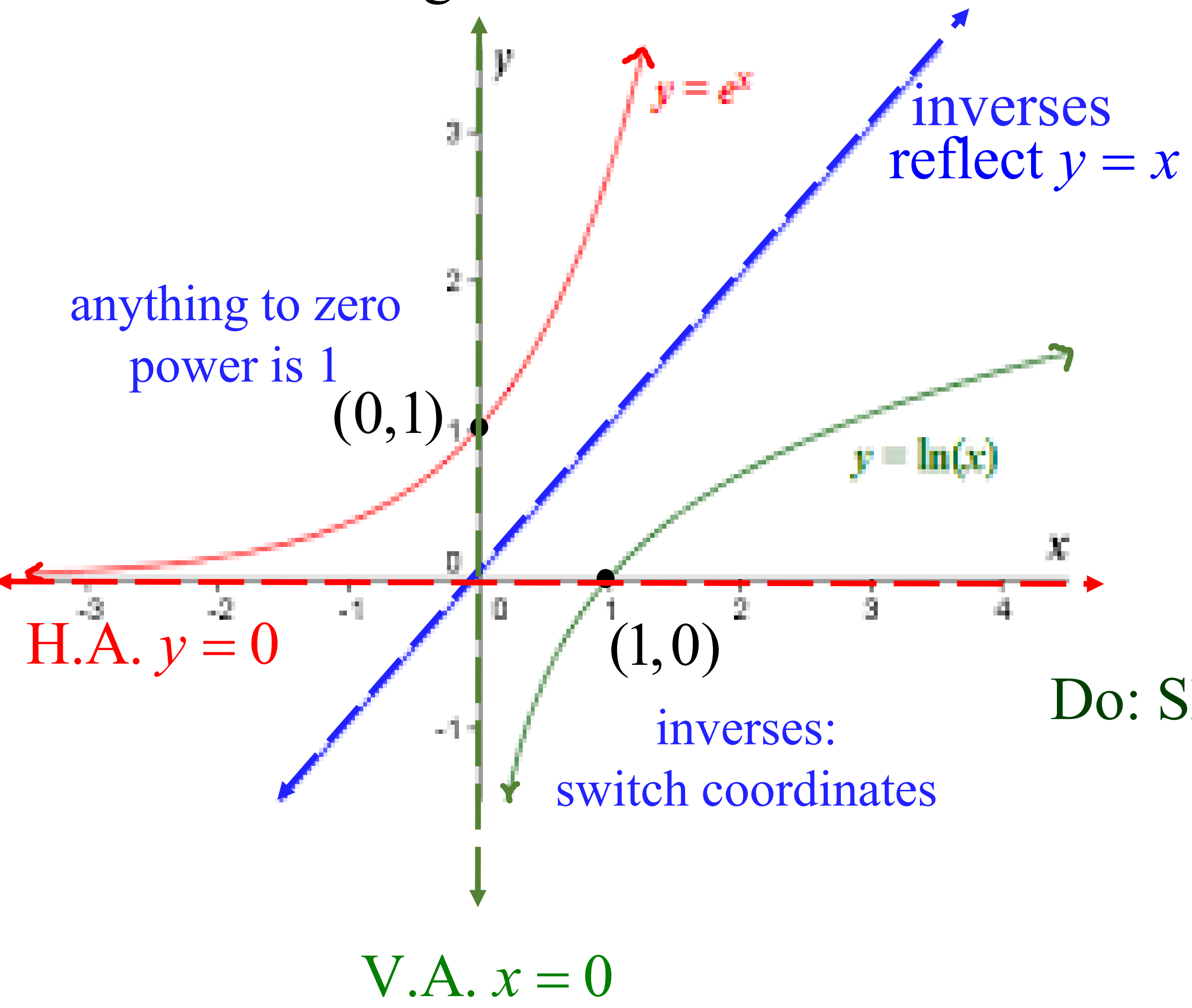
ex. Evaluate $\cancel{6}^{\cancel{\log_6} 9} = 9$

$\boxed{\ln e^x = x}$

$\boxed{e^{\ln x} = x}$

Logarithmic Graph

Log graph is inverse of the exponential graph
with correlating base.



$\underline{b^x}$	$\underline{\log_b x}$
domain: $\mathbb{R}$	domain: $(0,\infty)$
range: $(0,\infty)$	range: $\mathbb{R}$
asymptote: $y = 0$	asymptote: $x = 0$

$f(x)$

Do: Sketch graph $f(x) = \log_2(x-1)$.

Logarithm Rules

More complex logarithms can be expressed in one of two formats:

- *condensed*
- *expanded*

Use these rules to help evaluate equations with logs.

<u>Condensed Format</u>

Product Rule $\log_b(MN)$ multiplication inside parentheses

Quotient Rule $\log_b\left(\frac{M}{N}\right)$ division inside parentheses

Power Rule $\log_b M^P$ variable has an exponent

Product Rule

Product Rule: $\log_b(MN) = \log_b(M) + \log_b(N)$

condensed format — expanded format

ex. expand $\ln(xy) = \boxed{\ln x + \ln y}$

ex. expand $\log_4(7 \cdot 4) = \log_4 7 + \cancel{\log_4 4}^{1} = \boxed{\log_4 7 + 1}$

ex. expand $\log(10z) = \cancel{\log 10}^{1} + \log z = \boxed{1 + \log z}$

$\log 10 = \log_{10} 10$

ex. condense $\log_3 p + \log_3 q = \boxed{\log_3(pq)}$

ex. condense $\log_4 z + \log_4 y^7 + \log_4 5 = \boxed{\log_4\left(5y^7 z\right)}$

Quotient Rule

Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$

ex. expand $\log_7\left(\frac{19}{x}\right) = \boxed{\log_7 19 - \log_7 x}$

doesn't simplify

ex. expand $\ln\left(\frac{e^3}{7}\right) = \ln e^3 - \ln 7 = \boxed{3 - \ln 7}$

cancels

ex. condense $\log_3 x^4 - \log_3 \sqrt{y} = \boxed{\log_3\left(\frac{x^4}{\sqrt{y}}\right)}$

Power Rule

Power rule allows the exponent of the variable to become the log's coefficient and vice versa.

Power Rule: $\log_b M^P = P\log_b(M)$

note:
$\left(\log_b M\right)^P \neq \log_b M^P$

ex. expand $\ln x^2 = \boxed{2\ln x}$

ex. expand $\log_5 7^4 = \boxed{4\log_5 7}$

ex. expand $\ln 3^x = \boxed{x\ln 3}$

ex. expand $\ln\sqrt{x} = \ln x^{1/2} = \boxed{\frac{1}{2}\ln x}$

$\sqrt[n]{b} = b^{1/n}$

Expand Logarithmic Expressions

ex. Expand the following as much as possible:

- $\log_b\left(x^2\sqrt{y}\right) = \log_b x^2 + \log_b \sqrt{y}$

  PRODUCT

  $= 2\log_b x + \log_b y^{1/2}$

  $= \boxed{2\log_b x + \frac{1}{2}\log_b y}$

- $\log_6\left(\frac{\sqrt[3]{x}}{36y^4}\right) = \log_6 \sqrt[3]{x} - \log_6(36y^4)$

  QUOTIENT

  PRODUCT

  $= \log_6 x^{1/3} - (\log_6 36 + \log_6 y^4)$

  $= \boxed{\frac{1}{3}\log_6 x - \quad 2 \quad - 4\log_6 y}$

  $\log_6 36 = y$

  $6^y = 36$

  $y = 2$

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

$$\log_b M^P = P\log_b(M)$$

Do: Expand (and evaluate where applicable):

$$\log_5\left(\frac{\sqrt{x}}{125y^3}\right)$$

Condense Logarithmic Expressions

ex. Write the following as a single logarithm (or value):

- $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$

  PRODUCT

  $= \log_4(64)$

  $\boxed{= 3}$

  $\log_4 64 = y$

  $4^y = 64$

  $y = 3$

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

$$\log_b M^P = P\log_b(M)$$

- $\log(4x-3) - \log x = \boxed{\log\left(\frac{4x-3}{x}\right)}$

- $3\ln(x+7) - 4\ln x = \ln(x+7)^3 - \ln x^4$ 2. QUOTIENT

  1. POWER*

  $= \boxed{\ln \frac{(x+7)^3}{x^4}}$

*coefficients must be 1 to condense

Condense Logarithmic Expressions (cont'd)

ex. Write the following as a single logarithm:

- $4\log_b x + 2\log_b 6 - \frac{1}{2}\log_b y$

$= \underbrace{\log_b x^4 + \log_b 6^2}_{\text{PRODUCT}} - \log_b y^{1/2}$

$= \underbrace{\log_b(36x^4) \qquad -\log_b \sqrt{y}}_{\text{QUOTIENT}}$

convert exponents to radicals in condense

$= \boxed{\log_b\left(\frac{36x^4}{\sqrt{y}}\right)}$

final coefficient of a condensed logarithm will be **1**

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

$$\log_b M^P = P\log_b(M)$$

Condense Logarithmic Expressions - Do

Do: Write the following as a single logarithm:

- $5\log_6 x + 6\log_6 y$
- $4\ln x - 7\ln y + 3\ln z$
- $\frac{1}{3}\left(5\ln(x+6) - \ln(x^2 - 36)\right)$

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

$$\log_b M^P = P\log_b(M)$$

Change of Base

The main purpose to change the base of a logarithm is to work with the limitation of calculators.
(most calculators only offer base 10 and base e)

Change to Common Log: $\boxed{\log_b M = \dfrac{\log M}{\log b}}$

original base

Change to Natural Log: $\boxed{\log_b M = \dfrac{\ln M}{\ln b}}$

ex. Evaluate $\log_7 19$ correct to *two* decimal places.

$$= \frac{\ln M}{\ln b} = \frac{\ln 19}{\ln 7} \approx 1.51314... \qquad \boxed{A:\ 1.51}$$

Do: Evaluate $\log_6 17$ correct to *two* decimal places. $\boxed{A:\ 1.58}$

ex. Write as a single term that does not contain a logarithm: $e^{\ln 8x^5 - \ln 2x^2}$ condense

$$= \cancel{e}^{\cancel{\ln}\frac{8x^5}{2x^2}}$$

$$= \frac{8x^5}{2x^2} = \boxed{4x^3}$$